

Fairness and Efficiency in Online Advertising Mechanisms

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Abstract

Online advertising often involves targeting ads to certain types of consumers where ads are commonly sold by generalized second price auctions. However, such an auction or mechanism could be considered unfair if similar consumers are consistently shown different ads or consistently receive different payoffs. Results show that such ascending bid auctions may result in unfair treatment and additionally that uncertainty regarding an ad's value can result in inefficiency. An alternative way to assign ads to consumers is presented called the random assignment mechanism. Results show that the random assignment can improve fairness while improving efficiency in some circumstances.

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1 Introduction

The fairness and efficiency of ascending bid auctions is examined in an online advertising framework where there is uncertainty regarding the ad's value to the consumer and where advertisers can target ads to certain consumers.¹ Results give conditions under which ascending bid auctions are (and are not) fair and efficient in such a framework. Specifically, we show that the targeting of ads can lead to unfair treatment where similar consumers are treated differently. Additionally, uncertainty regarding an ad's value can lead to inefficiency if a consumer's interest level in an ad is quite different from what an advertiser expects.

In an effort to improve fairness and/or efficiency we also consider a random assignment mechanism where advertisers bid for a chance to display an ad and all advertisers above a certain threshold are given an equal chance to display an ad to any consumer.² By design this mechanism does not allow an advertiser to target certain consumers. Results show that fairness is generally achieved by the random assignment mechanism in that similar consumers are shown or can expect to be shown similar ads. However, the random assignment mechanism has a similar inefficiency to the ascending bid auction due to uncertainty regarding the ad's value. Lastly, we also compare the a priori efficiency of the two mechanisms by comparing the probability that a mechanism assigns the efficient advertiser to show an ad given the uncertainty regarding ad values. Results show that the

¹Google has a remarketing campaign to help advertisers target previous visitors to their sites; see developers.google.com/adwords/api/docs/guides/remarketing for more information about remarketing. Such ads are sold using a bidding process similar to a generalized second price (gsp) auction.

²The idea of using randomization to address fairness is also used in the algorithmic classification literature of Dwork and Ilvento [2018] and Dwork, Hardt, Pitassi, Reingold, and Zemel [2012].

random assignment mechanism is more likely to assign the efficient advertiser if the consumer specific interest level has a large variation compared with the common ad quality component while the ascending bid auction is more likely to assign the efficient advertiser if the common ad quality component has a large variation or if there is an advertiser with a significantly higher common ad quality component than other advertisers.

The current paper is most closely related to Athey and Ellison [2011] and Jeziorski and Segal [2015]. Athey and Ellison [2011] consider sponsored link ad auctions where consumer search strategies are examined and show that a two-stage auction can increase efficiency in a model with click-weighting where interested consumers have similar ad values. Jeziorski and Segal [2015] provides an empirical analysis of sponsored search ads with a theory component where ads have a consumer specific value component; Jeziorski and Segal [2015] show that targeting ads can lead to a higher click through rate. Our analysis adds to this literature by considering the fairness of such auctions and shows that targeting consumers can lead to unfair treatment of similar consumers. Additionally, we show that inefficiency can result even if consumers have a high click through rate if the value maximizing ad is not shown.

Other related literature includes the growing literature on sponsored link auctions examining issues such as exclusive display (Sayedi, Jerath, and Baghaie [2018]), position ranking (Chu, Nazerzadeh, and Zhang [2020]), and experimental studies of online auctions (Sun, Fan, and Tan [2020]), as well as the classic generalized second price auction literature of Varian [2007], Varian and Harris [2014], and Edelman, Ostrovsky, and Schwarz [2007]. There is also a literature examining online shopping with search costs (Choi, Dai, and Kim [2018]) as well as consumer

search for the best online price (Wang and Wright [2020]). The targeting of ads has privacy issues regarding the disclosure of consumer information to advertisers; see De Corniere and De Nijs [2016]. A literature review of privacy issues is presented by Acquisti, Taylor, and Wagman [2016].

Additionally, our paper is related to the economics fairness literature of Rawls [1971] and Roemer [1986]; see also the literature reviews of Fleurbaey and Maniquet [2018] and Roemer and Trannoy [2016]. The current paper is also related to the fairness literature focusing on algorithmic fairness; see Dwork and Ilvento [2018], Dwork, Hardt, Pitassi, Reingold, and Zemel [2012], and Lepri, Oliver, Letouzé, Pentland, and Vinck [2018].

2 Model

There is a continuum of consumers who visit a website and observe m displayed advertisements. Consumer i can click on any of the ads displayed at a cost of s_i per click. Assume that each s_i is a random variable independently and identically distributed on $[0, \bar{s}]$ with a continuous distribution H . Additionally, assume s_i is private information known only to i .

There are $n > m$ advertisers who want to advertise on the website. The value of clicking on advertiser j 's ad to consumer i is $v_{ji} = v_j + z_{ji}$, where v_j represents a quality component common to everyone for j 's ad and z_{ji} represents a consumer specific quality component or interest level for j 's ad. Assume that each v_j and z_{ji} are random variables independently and identically distributed from common distributions F and G , respectively. Let v_j and z_{ji} have respective supports $[\underline{v}, \bar{v}]$ and $[\underline{z}, \bar{z}]$, where $\underline{v} \geq 0$ and $\underline{z} \geq 0$. Additionally, assume that initially each

advertiser j knows v_j , but only knows the distribution and support of z_{ji} . And initially each consumer i only knows the distribution and support of v_j and z_{ji} . However, if i clicks on j 's ad then i learns v_{ji} . As in Athey and Ellison [2011], assume that if a consumer clicks on the ad, then the probability that the advertiser satisfies the consumer's need is v_{ji} , and if the need is met the advertiser receives a payoff of 1. As v_{ji} can be interpreted as a probability we assume that $\bar{v} + \bar{z} \leq 1$ and that $\underline{v} + \underline{z} \geq 0$.

Let time be represented by $T = \{1, 2, \dots, t, \dots\}$. Each period t the advertisers are given the chance to bid on the m displayed ads for each person i . We consider two different mechanisms for matching ads to consumers, the ascending bid auction and the random assignment mechanism.

First, we define the *ascending bid auction*. As in a simplified version of Athey and Ellison [2011] and Edelman, Ostrovsky, and Schwarz [2007], the website runs an auction for the ads displayed to i . Let the auction start at zero and rise with a clock. Each advertiser then decides when to drop out. Let j drop out of the auction at price b_{ji}^t . The auction ends when there is only one bidder left. Let the top m bidders each display an ad to i and let each pay the $m + 1$ highest price per click. If there is no $(m + 1)$ bid, then set this bid to 0. Define $q_{ji}^t = 1$ if j is one of the top m bidders to display an ad to i and let $q_{ji}^t = 0$ otherwise. We can interpret q_{ji}^t as the probability that j 's ad is shown to i .

Note that the ascending bid auction allows bidding for ads targeted at certain consumers such as those who have clicked on an ad previously or those with certain characteristics.

Second, we define the *random assignment mechanism*. Let each advertiser j submit one bid b_j^t for displaying an ad to any consumer in period t . Any advertiser

who submits a bid above a floor of α is given an equal chance to display an ad to consumer i at price α where bids are only paid if the ad is displayed. Here α is chosen from $[\underline{\alpha}, \bar{\alpha}]$ by the website in period 1. Let \hat{j} be the number of advertisers with $b_j^t \geq \alpha$. We define $q_{ji}^t = \frac{1}{\hat{j}}$ for all j with $b_j^t \geq \alpha$ and $q_{ji}^t = 0$ otherwise. Thus, q_{ji}^t represents the probability that j 's ad is shown to i .

Note that this mechanism does not allow an advertiser to make a bid to a specific consumer or type of consumer, but rather every period the advertiser makes one bid for display to any consumer. For $m > 1$ we assume ads are displayed to i in a random order.

In period t , let the ads displayed to i be represented by M_i^t . Let the ads clicked on by i be represented by \tilde{M}_i^t with cardinality $|\tilde{M}_i^t|$. Then i 's period t payoff is represented by $u_i^t = \sum_{j \in \tilde{M}_i^t} v_{ji} - |\tilde{M}_i^t| s_i$. And i 's payoff from clicking on j 's ad is represented by $u_{ji}^t = v_{ji} - s_i$.

We define an allocation of ads to consumers to be *efficient* if the sum of consumer payoffs are maximized. As each ad can be displayed to multiple consumers and a different set of ads can be displayed to each consumer the allocation is efficient if each consumer's payoff is maximized. Assume that for each i there exists m_i advertisers such that $v_{ji} \geq s_i$. Let then, an allocation, M^t , is *efficient*, if the ads displayed to each user maximize his payoff or if $M_i^t = \arg \max_{\hat{M} \in \mathcal{M}_i} \sum_{j \in \hat{M}} v_{ji}$ for all i . Here \mathcal{M}_i represents all possible sets of ads of size $\min\{m, m_i\}$; thus, ads not worth clicking on for i will never be efficient to display. Note that as only $\min\{m, m_i\}$ ads are shown to each i , s_i is not needed in our maximization problem. Additionally, notice that since the advertiser's expected payoff from meeting a consumer's need is also equal to v_{ji} our definition of efficient also maximizes the sum of the expected payoffs to advertisers.

We define the ad display system to be *fair*, if similar agents receive a similar time t payoff. Specifically, if there exists two agents i and \hat{i} such that $|v_{ji} - v_{j\hat{i}}| < \epsilon$ for all j and $|s_i - s_{\hat{i}}| < \epsilon$ where ϵ is a small positive number. Then the ad display system is *fair* if $E[u_i^t] - E[u_{\hat{i}}^t] < m \cdot \epsilon$. This definition of fairness is based on the resource egalitarian fairness of Rawls [1971], see also Fleurbaey and Maniquet [2018]. Additionally, this definition of fairness is related to Dwork and Ilvento [2018]’s notion of multiple task fairness which requires that similar individuals receive a similar expected outcome.

An alternative definition of fairness is that of *opportunity fairness* which states that similar agents should have similar opportunities or that if $|v_{ji} - v_{j\hat{i}}| < \epsilon$ for all j and $|s_i - s_{\hat{i}}| < \epsilon$ where ϵ is a small positive number. Then the ad display system is *opportunity fair* if $|q_{ji}^t - q_{j\hat{i}}^t| < \epsilon$. This fairness definition is related to Roemer [1993]’s equality of opportunity; see also Roemer [1998], and Roemer and Trannoy [2016]. Under opportunity fairness, if agents have similar preferences and click costs, then each ad should have a similar chance of being displayed to them. Note that this definition allows agents to click on different ads while the previous fairness definition requires that agents also expect to click on the same set of ads.

3 Results

We investigate the fairness and efficiency of the ascending bid auction and random assignment mechanism. First, consider the ascending bid auction.

Proposition 1 *Let $v_1 > v_2 > \dots > v_n$ and let $\underline{z} = \bar{z}$. Assume $v_m + \underline{z} \geq s_i$ where v_m is the m th largest v_j . Then the ascending bid auction is efficient and opportunity fair for all t and is fair at $t = 1$.*

If $\underline{z} = \bar{z}$, then all consumers have the same interest level for all ads and this interest level is known. Thus, the advertiser with the highest common quality component will also have the highest ad value for all consumers. The ascending bid auction will assign the advertisers with the m highest common quality components to the consumer's m ad slots. As the j th ad has the j th highest value, this allocation is efficient. This allocation is also opportunity fair as each consumer is shown the same set of ads.

Proof. As $\underline{z} = \bar{z}$, advertisers initial beliefs regarding consumer valuations will be correct as all consumers will have the same interest level for all ads and this interest level is known. In the ascending bid auction, j will be willing to bid up to $b_{ji}^t = v_j + \underline{z}$. The auction to display m ads to i will be won by the advertisers with the m highest v_j 's. As $v_m + \underline{z} \geq s_i$, it will always be beneficial for a consumer to click on one of these ads. There is no information for the advertiser to learn here regarding consumer preferences, so advertisers will not have incentive to change their bids in the future. Advertisers $\{1, 2, \dots, m\}$ will be the only advertisers to display an ad. This auction is efficient since $\sum_{j=1}^m (v_j + \underline{z})$ maximizes i 's payoff or $\sum_{j=1}^m (v_j + \underline{z}) = \max_{\hat{M} \in \mathcal{M}_i} \sum_{j \in \hat{M}} v_{ji}$ for all j and i . This auction is opportunity fair as all agents will have the same set of ads displayed to them.

This auction is fair if similar agents receive a similar expected payoff. Here all agents are displayed the same ads. We show that any two agents with similar click costs either both click on an ad, or both have expected utility within ϵ . Consider i and \hat{i} such that $|s_i - s_{\hat{i}}| < \epsilon$. Without loss of generality assume $s_i \leq s_{\hat{i}}$. Assume i clicks on a given ad or that $s_i \leq E[v_k] + \underline{z}$. If $s_{\hat{i}} \leq E[v_k] + \underline{z}$ then \hat{i} also clicks on this ad. If instead $s_{\hat{i}} > E[v_k] + \underline{z}$, then \hat{i} does not click on this ad. Consider the

expected payoffs of i and \hat{i} in this case, when $t = 1$. Here $E[u_{ji}^1] = E[v_j] + \underline{z} - s_i$ and $E[u_{\hat{j}\hat{i}}^1] = 0$. Thus, $E[u_{ji}^1] - E[u_{\hat{j}\hat{i}}^1] = E[v_j] + \underline{z} - s_i$. Since $s_i \leq E[v_k] + \underline{z} < s_{\hat{i}}$ and $s_{\hat{i}} - s_i < \epsilon$ it must be that $E[v_j] + \underline{z} - s_i < \epsilon$. As each consumer is shown at most m ads it must be that $E[u_i^t] - E[u_{\hat{i}}^t] < m \cdot \epsilon$. Thus, i and \hat{i} have similar expected payoffs and the auction is fair. \square

Proposition 1 assumes that z_{ji} is the same for all consumers and that this is known to the advertisers. If however advertisers' beliefs regarding z_{ji} are quite different from the actual z_{ji} , then the ascending bid auction may not be efficient or fair as the next set of propositions show.

Proposition 2 *Let $m = 1$. Let there exist advertisers j and k and consumer i such that $v_k = \max_{\ell} v_{\ell} > v_j$ and $z_{ki} - z_{ji} < v_j - v_k$. And assume $s_i < v_k + z_{ki}$ and $\max\{s_i, E[s_i]\} < E[v_k] + E[z_{ki}]$. Then the ascending bid auction is not efficient.*

In Proposition 2, advertiser k has the largest common quality component and wins the ascending bid auction for consumer i . When i is shown k 's ad he learns it is worth clicking on and will continue to click on it in future periods. This act of clicking will reinforce k 's beliefs that i values his ad and k will continue to bid the most. However, consumer i would receive a larger value from j 's ad as z_{ji} is quite large; but as i is never shown j 's ad the value of z_{ji} is not learned. Thus, the ascending bid auction will result in an inefficient ad allocation.

Proof. At $t = 1$, each advertiser ℓ will set $b_{\ell i}^1 = v_{\ell} + E[z_{\ell i}]$. Advertiser k will win the auction as $E[z_{\ell i}]$ is the same for all ℓ and as $v_k = \max_{\ell} v_{\ell}$. Since $s_i < E[v_k] + E[z_{ki}]$, i will click on k 's ad. As $s_i < v_k + z_{ki}$, after i clicks on the ad he will learn that clicking on the ad was worth while as expected and will click on the ad again in future periods. From advertiser k 's perspective, $E[s_i] < E[v_k] + E[z_{ki}]$ so he

expected i to click on the ad and i does. Advertiser k 's beliefs are confirmed as correct and he will not update his beliefs in period 1. In period 2, k will win the auction again and i will click on the ad. Now k will learn from i 's click that $s_i < v_k + z_{ki}$ and k will update $E[v_{ki}]$ upwards to $E[v_{ki}|s_i < v_k + z_{ki}]$. In future periods, k will continue to win the auction and i will continue to click.

As k receives a payoff of 1 only if he meets i 's need and this occurs with probability v_{ki} , it is possible with enough clicks k will be close to learning v_{ki} . Regardless, k continues to win the auction. However, since $z_{ki} - z_{ji} < v_j - v_k$ implies $v_j + z_{ji} > v_k + z_{ki}$, i would be better off if j won the auction. Thus, this auction is inefficient. \square

Let \hat{i} be a clone of i , if $s_i = s_{\hat{i}}$ and $z_{ji} = z_{j\hat{i}}$ for any j .

Proposition 3 *Let $m = 1$. Let there exist advertiser k and consumer i such that $v_k = \max_{\ell} v_{\ell}$ and such that $s_i < v_k + z_{ki}$ and $E[s_i] < E[v_k] + E[z_{ki}]$. In period $t = 3$, let a new advertiser \hat{k} and a new consumer \hat{i} enter the market where $v_{\hat{k}} = v_k + \beta$ and \hat{i} is a clone of i . Then the ascending bid auction is neither opportunity fair nor fair in periods $t \geq 3$ for $\beta > 0$ small enough.*

Here k wins the auction to display an ad to i initially. As i always clicks on k 's ad, k also learns that i values the ad and will update upwards his expectation of z_{ki} . Thus, at $t = 3$, k will also update upwards his bid to display the ad to i . When the new advertiser \hat{k} enters at $t = 3$, he will win the auction to display the ad to \hat{i} as $v_{\hat{k}} = v_k + \beta$. However, if β is small enough, then k will continue to win the auction to display to i . As i and \hat{i} are identical this auction allocation is neither fair nor opportunity fair since they will be shown different ads.

Proof. As $v_k = \max_{\ell} v_{\ell} > v_j$, advertiser k will win the auction to display the

ad to consumer i in all periods. From the proof of Proposition 1, we know that after i clicks on k 's link twice, k will update his beliefs regarding $E[v_{ki}]$ from $E[v_{ki}] = v_k + E[z_{ki}]$ to $E[v_{ki}|s_i < v_k + z_{ki}] = v_k + E[z_{ki}|s_i < v_k + z_{ki}]$. Next we show that $v_k + E[z_{ki}] \leq v_k + E[z_{ki}|s_i < v_k + z_{ki}]$ or that $E[z_{ki}] \leq E[z_{ki}|s_i < v_k + z_{ki}]$. If $\bar{s} < v_k + \underline{z}$, then it is always true that $s_i < v_k + z_{ki}$ and therefore $E[z_{ki}] = E[z_{ki}|s_i < v_k + z_{ki}]$. Now consider the case where $\bar{s} \geq v_k + \underline{z}$. After i clicks on k 's link in period 2, k learns that $s_i < v_k + z_{ki}$. For any given s_i , k expects that $v_k + E[z_{ki}] > s_i$. Thus, if $s_i = v_k + \underline{z} + \epsilon$ then k knows that $v_k + z_{ki} \notin [v_k + \underline{z}, v_k + \underline{z} + \epsilon]$, and $v_k + z_{ki}$ must instead be above this range. Similar analysis holds for larger s_i resulting in k 's expectation of z_{ki} increasing to $E[z_{ki}|s_i < v_k + z_{ki}] \geq E[z_{ki}]$.

In period 3, let a new advertiser, say \hat{k} , enter the market. Let this advertiser have $v_{\hat{k}i}$ with $v_{\hat{k}} = v_k + \beta$ where $0 < \beta < E[z_{ki}|s_i < v_k + z_{ki}] - E[z_{ki}]$. Advertiser k will still win the auction to display the ad to i as the max k is willing to bid is $v_k + E[z_{ki}|s_i < v_k + z_{ki}]$ which is greater than $v_{\hat{k}} + E[z_{\hat{k}i}] = v_{\hat{k}} + E[z_{ki}]$ which is the max \hat{k} is willing to bid. Now let a clone of person i , say \hat{i} , enter the market in period 3. Advertiser \hat{k} will win the auction to display an ad for \hat{i} as now what k is willing to bid, $v_k + E[z_{ki}]$ is less than $v_{\hat{k}} + E[z_{\hat{k}i}] = v_{\hat{k}} + E[z_{ki}]$ which is what \hat{k} is willing to bid. Thus, i and \hat{i} will always view different ads and will receive different payoffs. Note that the payoffs i and \hat{i} expect are also different as i expects to be shown k 's ad which he values at v_{ki} and \hat{i} is initially unsure which ad he will see, but will eventually expect to see \hat{k} 's ad with payoff $v_{\hat{k}i}$. Thus, the mechanism is not fair as the expected payoffs are different and is not opportunity fair as these agents expect to see different ads. In periods $t > 3$ a similar result will occur. The only difference is that if with enough periods or clicks k will be close to learning the true z_{ki} . However, his estimate should remain biased upwards, so for ϵ small

enough the auction will continue to be both unfair and opportunity unfair for players i and \hat{i} . \square

Next we consider the random assignment mechanism of assigning advertisers to consumers to see if efficiency and fairness can be improved.

Proposition 4 *The random assignment mechanism is opportunity fair for all t and is fair at $t = 1$.*

Opportunity fairness implies that similar agents expect to be shown the same set of ads while fairness implies that they also expect the same payoff. Even though similar agents expect to be shown the same ads, they may actually see different ads. Thus, similar agents can learn different information about which ads are worth clicking on. In later periods, similar agents may choose to click on different ads based on this information which will affect both future payoffs and future payoff expectations. Thus, fairness is guaranteed initially while opportunity fairness is always guaranteed.

Proof. First, we show that under the random assignment mechanism is opportunity fair or that similar individuals will expect to receive similar opportunities in all periods. Consider i and \hat{i} such that $|v_{ji} - v_{j\hat{i}}| < \epsilon$ and $|s_i - s_{\hat{i}}| < \epsilon$. We show that such agents will expect to have the same ads displayed to them. Let advertisers $\{1, 2, \dots, j, \dots, \bar{j}\}$ be the only advertisers with $b_j^t \geq \alpha$. Then each of these advertisers has an equal chance to display an ad to i and \hat{i} . If $\bar{j} \geq m$, then $q_{ji}^t = q_{j\hat{i}}^t = \frac{m}{\bar{j}}$ for $j \in \{1, 2, \dots, j, \dots, \bar{j}\}$. If $\bar{j} < m$, then $q_{ji}^t = q_{j\hat{i}}^t = 1$ for $j \in \{1, 2, \dots, j, \dots, \bar{j}\}$. And $q_{ji}^t = q_{j\hat{i}}^t = 0$ for $j \in \{\bar{j} + 1, \dots, n\}$. Therefore, i and \hat{i} both expect to see the same ads and the random assignment mechanism is opportunity fair.

Second, we show that the mechanism is fair at $t = 1$ or that initially i and \hat{i} expect to receive the same payoff. We already showed that each $j \in \{1, 2, \dots, j, \dots, \bar{j}\}$ has an equal chance of being shown to i and \hat{i} . We next show that the expected payoff difference between i and \hat{i} for each such j is less than ϵ ; which implies that $E[u_i^t] - E[u_{\hat{i}}^t] < m \cdot \epsilon$. Consider $j \in \{1, 2, \dots, j, \dots, \bar{j}\}$. If j 's ad is shown, then there are three possible actions by i and \hat{i} . The first is that both i and \hat{i} click on j 's ad. In this case i 's expected payoff is $E[v_{ji}] - s_i$ and \hat{i} 's expected payoff is $E[v_{j\hat{i}}] - s_{\hat{i}}$. As $E[v_{ji}] = E[v_{j\hat{i}}]$ it must be that $|E[v_{ji}] - s_i - E[v_{j\hat{i}}] + s_{\hat{i}}| = |s_{\hat{i}} - s_i| < \epsilon$. The second case is that neither i nor \hat{i} click on j 's ad in which case each consumer expects a payoff of 0 from the interaction. The third case is that one of the consumers clicks on the ad and the other does not. Without loss of generality, let $s_i < E[v_{ji}] = E[v_{j\hat{i}}] < s_{\hat{i}}$ and so i clicks on j 's ad while \hat{i} does not. Consumer i 's expected payoff is $E[v_{ji}] - s_i$ and \hat{i} 's expected payoff is 0. The difference between the payoffs is $E[v_{ji}] - s_i$. Since $s_i \leq E[v_{ji}] < s_{\hat{i}}$ and $s_{\hat{i}} - s_i < \epsilon$ it must be that $E[v_{ji}] - s_i < \epsilon$. Thus, i and \hat{i} have similar expected payoffs and the auction is fair at $t = 1$. \square

Let $v_1 > v_2 > \dots > v_n$ and let m be the advertiser with the m th highest common quality component.

Proposition 5 *Let $\underline{z} = \bar{z}$. The random assignment mechanism is efficient for all t , if $v_m + \underline{z} \geq \alpha > v_{m+1} + \underline{z}$ and if $v_m + \underline{z} \geq s_i$ for all i .*

If all consumers have the same interest level for all ads, then the random assignment mechanism is efficient if α is chosen so that only those advertisers with the top m quality components are willing to submit a bid that is above this floor. These ads will be the only ones shown which is efficient.

Proof. As $\underline{z} = \bar{z}$, advertisers initial beliefs regarding consumer valuations will be correct. As $v_m + \underline{z} \geq \alpha > v_{m+1} + \underline{z}$ only advertisers $\{1, 2, \dots, m\}$ will submit bids high enough to participate in the auction, as k will submit a bid of $v_k + \underline{z}$. This mechanism is efficient as it guarantees all consumers will be shown ads $\{1, 2, \dots, m\}$ and these are the ads that all consumer's value the most. Additionally, these ads are also worth clicking on for all i since $v_m + \underline{z} \geq s_i$. \square

Note that if $\alpha \leq v_{m+1} + \underline{z}$, then more than m advertisers will have bids greater than or equal to α and these advertisers will all have the same chance to display an ad to a consumer. Since each consumer values the top m ads the most, this allocation will not always be efficient. If instead $\alpha > v_m + \underline{z}$, then advertiser m will no longer submit a bid above α and so consumers will be shown less than the top m ads which is also not efficient. As the next proposition illustrates, there are also other conditions under which the random assignment mechanism is inefficient.

Continue to let $v_1 > v_2 > \dots > v_n$.

Proposition 6 *Let $m = 1$ and let $z_{1i} - z_{2i} < v_2 - v_1$. Assume $v_1 + E[z_{1i}] > \alpha > v_2 + E[z_{2i}]$ and $\max\{s_i, E[s_i]\} < v_1 + z_{1i}$ and $\max\{s_i, E[s_i]\} < v_1 + E[z_{1i}]$ for all i . Then the random assignment mechanism is not efficient.*

Proof. As $s_i < v_1 + z_{1i}$ and $v_1 + E[z_{1i}] > \alpha > v_2 + E[z_{2i}]$, only advertiser 1 will submit a bid above α and 1's ad will be shown to all consumers. As $z_{1i} - z_{2i} < v_2 - v_1$, consumer i would be better off if 2's ad were shown. Thus, this allocation is not efficient. Note that as $\max\{s_i, E[s_i]\} < v_1 + z_{1i}$ and $\max\{s_i, E[s_i]\} < v_1 + E[z_{1i}]$ for all i , all consumers will click on 1's ad and this is also what 1 expects to have happen so 1 will not change his bid in future periods. \square

Note that the conditions of Proposition 6 are quite similar to that of Proposition 2. Thus, both mechanisms will assign the incorrect advertiser if the advertiser with the largest common quality component is not the advertiser with the largest value to the consumer.

In the next proposition we calculate the probability that the efficient advertiser is assigned to display an ad to i by each mechanism. Recall that each z_{ji} is randomly drawn from distribution G and that advertisers do not initially have any additional information regarding z_{ji} .

Let \tilde{n} represent the number of advertisers willing to bid in the random assignment mechanism or the number with $v_j + E[z_{ji}] \geq \alpha$.

Proposition 7 *Let $m = 1$ and $s_i = 0$ for all i . Let $v_k = \max_{\ell} v_{\ell}$ and assume $v_k \geq \alpha - \max\{z_{ki}, E[z_{ki}]\}$. Then the probability that the efficient advertiser is assigned to display an ad for consumer i equals $\prod_{j \neq k} \left(\int_{\underline{z}}^{\max\{\underline{z}, \bar{z} - (v_k - v_j)\}} G(v_k - v_j + z)g(z)dz + 1 - (G(\max\{\underline{z}, \bar{z} - (v_k - v_j)\})) \right)$ under the ascending bid auction mechanism and the probability equals $1/\tilde{n}$ under the random assignment mechanism.*

For the random assignment mechanism, there are \tilde{n} advertisers willing to bid above α and each will be given an equal chance to display an ad to i . Thus the probability that the efficient ad is shown to i is $1/\tilde{n}$. For the ascending bid auction, advertiser k has the largest common component and will therefore win the auction to display the ad to i . However, if there exists advertiser j with a larger value $v_{ji} > v_{ki}$, then efficiency will require j to display the ad not k . The probability of this event occurring is given in the proposition.

Proof. First, consider the random assignment mechanism. Each advertiser j will submit bid $b_j^t = v_j + E[z_{ji}]$ as this equals j 's expected payoff of advertising to i . By

assumption there are \tilde{n} advertisers with $v_j + E[z_{ji}] \geq \alpha$. At the random assignment mechanism gives each advertiser who submits a bid above α an equal chance to display an ad, each advertiser with $b_j^t \geq \alpha$ has a chance of $1/\tilde{n}$. As $v_k + z_{ki} \geq \alpha$ it must also be that $v_{\hat{k}} + z_{\hat{k}i} \geq v_k + z_{ki} \geq \alpha$ where \hat{k} is the ad that maximizes i 's payoff. Thus, the probability that the ad which maximizes i 's payoff is displayed to i is $1/\tilde{n}$. Note that as $s_i = 0$, all agents click on all ads; thus, agents do not learn that some ads are less valuable than the clicking cost. Advertiser j knows this and will have no incentive to update his bid. Thus, the probability that the correct ad is displayed is $1/\tilde{n}$ for all t .

In the ascending bid auction, k will win the auction to display an ad to i as $v_k = \max_{\ell} v_{\ell}$. Next we compute the probability that k maximizes i 's payoff and thus is the efficient advertiser to display an ad to i . Advertiser k maximizes i 's payoff if $v_{ki} \geq v_{ji}$ for all $j \neq i$ or $v_k - v_j \geq z_{ji} - z_{ki}$. The probability that $v_k - v_j \geq z_{ji} - z_{ki}$ for a given $v_k - v_j$ equals $\int_{\underline{z}}^{\max\{\underline{z}, \bar{z} - (v_k - v_j)\}} Pr(z_{ji} - z_{ki} < v_k - v_j | z_{ki} = z)g(z)dz + 1 - (G(\bar{z} - (v_k - v_j)))$ or $\int_{\underline{z}}^{\max\{\underline{z}, \bar{z} - (v_k - v_j)\}} G(v_k - v_j + z)g(z)dz + 1 - (G(\max\{\underline{z}, \bar{z} - (v_k - v_j)\}))$. As each z_{ji} is identically and independently distributed the probability that all $j \neq k$ have $v_{ji} \leq v_{ki}$ is $\prod_{j \neq k} (\int_{\underline{z}}^{\max\{\underline{z}, \bar{z} - (v_k - v_j)\}} G(v_k - v_j + z)g(z)dz + 1 - (G(\max\{\underline{z}, \bar{z} - (v_k - v_j)\})))$. \square

In the following example we compare the probability of assigning the efficient advertiser by the random assignment mechanism to that of the ascending bid auction. The example illustrates that the random assignment mechanism is more likely to choose the efficient advertiser when the consumer specific quality component has a large variation and the common quality components are somewhat similar; here it is less likely that the advertiser chosen by the ascending bid auction

who has the largest common quality will also be efficient. While the ascending bid auction is more likely to choose the efficient advertiser if there is an advertiser whose common quality component is significantly larger than the other common components as this advertiser is likely to be efficient and will always be chosen by the ascending bid auction.

Example. Let $z_{ji} \sim U[0, .5]$, $n = 3$, $m = 1$, $(v_1, v_2, v_3) = (.4, .35, .35)$, $\alpha = .3$, and let $s_i = 0$ for all i . First, consider the ascending bid auction. As advertiser 1 has the largest v_j , 1 will have the highest bid and will win the auction to display the ad to i . However, it will only be efficient for 1 to win the auction if $v_{1i} > v_{2i}$ and $v_{1i} > v_{3i}$. From Proposition 7, the probability that $v_{1i} > v_{2i}$ given $v_1 - v_2 = .05$ equals $Pr(z_{2i} - z_{1i} < .05) = \int_0^{.45} \frac{.05+z}{.5} \cdot \frac{1}{.5} dz + \frac{.05}{.5} = .55$. And $Pr(z_{3i} - z_{1i} < .05) = .55$. Thus, the probability that both these events occur is $(.55)(.55) = .30$. Now we compare this probability to that of the random assignment mechanism. By Proposition 7, as all advertisers have $v_j + E[z_{ji}] = v_j + .25 \geq \alpha = .3$ the probability that the efficient advertiser is assigned is $\frac{1}{3} > .30$. Here advertisers have similar common quality components and the consumer specific quality component has a large variation; thus even though advertiser 1 has the largest v_j he may not have the largest v_{ji} and thus may not be the efficient ad choice. As a result, the random assignment mechanism has a larger probability of assigning the efficient advertiser to display an ad than the ascending bid auction which always chooses 1 to display the ad.

We also investigate what happens to these probabilities as the number of advertisers increases. First, we increase n to $n = 5$ by setting $(v_1, v_2, v_3, v_4, v_5) = (.4, .35, .35, .35, .35)$. Now in the ascending bid auction the probability that the ef-

efficient advertiser is chosen decreases from .3 to $(.55)^4 = .09$ while with the random assignment mechanism the probability decreases from $\frac{1}{3}$ to $\frac{1}{5}$. Thus, the decrease is more severe in the ascending bid auction case as the advertisers still have quite similar common quality components while the consumer specific quality has a large variation making it less likely that 1 has the efficient ad.

Next we investigate how changing the common quality component affects this probability. Leave everything the same in the initial example except change (v_1, v_2, v_3) to $(.4, .2, .1)$. Thus, it should be more likely that ad 1 is efficient or has the largest value since the common quality component for the other ads has decreased, but the consumer specific quality has the same distribution. Again we consider the ascending bid auction first. Now $Pr(v_{1i} > v_{2i} | v_1 - v_2 = .4 - .2) = Pr(z_{2i} - z_{1i} < .2) = .82$ and $Pr(z_{3i} - z_{1i} < .3) = .96$. Thus, the probability that both events occur is .79. If instead we consider the random assignment mechanism, then the probability that the efficient advertiser displays the ad is $\frac{1}{3}$. Thus, now the ascending bid auction has a higher probability of choosing the efficient advertiser. Here advertiser 1 has a common quality component that is significantly above the other common components which makes it more likely that he is the efficient advertiser. As 1 is always chosen by the ascending bid auction it is more likely that this auction chooses the efficient advertiser.

We also investigate what happens if we increase n to $n = 5$ by setting $(v_1, v_2, v_3, v_4, v_5) = (.4, .2, .2, .2, .1)$. Now the probability that ad 1 is shown instead of any other ad decreases from .79 to $(.82)^3 \cdot (.96) = .53$ in the ascending bid auction. With the random assignment mechanism the probability decreases from $\frac{1}{3}$ to $\frac{1}{5}$. Again the decrease is more severe with the ascending bid auction. Here as the number of advertisers grows it becomes less likely that a given advertiser is chosen. In fact

if the number of advertisers increases to $n = 22$ by adding more advertisers with $v_j = .2$ then the probability that the efficient ad is chosen in the ascending bid auction case decreases to $(.83)^{20} \cdot (.96) = .018$ while in the random assignment mechanism it decreases to $\frac{1}{22} = .045$. Thus, even if advertiser 1 has a significantly higher common component the probability that 1 has the efficient ad decreases quite a bit if the number of other advertisers increases by a large amount and so now the ascending bid auction has a lower probability of assigning the efficient ad.

4 Conclusion

The fairness and efficiency of ascending bid auctions and a random assignment mechanism are examined in an online advertising framework. Results show that the random assignment mechanism can increase fairness and may increase efficiency if the consumer specific interest level has a large variation compared to the common quality component.

Possible extensions include examining group fairness. Our fairness notions ask that similar people have similar opportunities or similar payoffs. Group fairness would ask that groups of similar people are treated equally; see Dwork and Ilvento [2018], Dwork, Hardt, Pitassi, Reingold, and Zemel [2012]. In our framework, one might ask that a group of similar people have an equal chance to see a valuable ad as another group. One could then compare the ascending bid auction and the random assignment mechanism in terms of group fairness. We leave such an analysis for future research.

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